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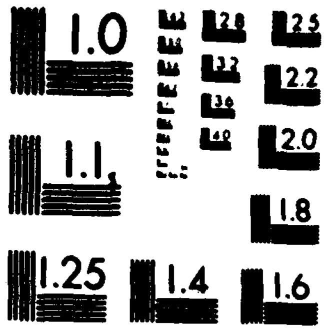

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ALL SPECTRAL DOMINANT NORMS ARE STABLE

S. Friedland* and C. Zenger**

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ABSTRACT

A vector norm $|\cdot|$ on the space of $n \times n$ complex valued matrices is called stable if

$$|A^m| \le K|A|^m$$

for all A and non-negative integers m. We show that such a norm is stable if and only if it dominates the spectral radius.

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SIGNIFICANCE AND EXPLANATION

When solving partial differential equations numerically one often has to use iterations involving matrices restricted to a given set A of $n \times n$ complex valued matrices. It then follows that the iteration scheme is stable if and only if this set of matrices is stable. That is all powers of all matrices from the set A are uniformly bounded. Such sets were completely characterized by H. O. Kreiss. However, his criteria are hard to use.

In this paper ~~we~~ the authors characterize ~~in~~ a very simple way stable sets of matrices A, whenever the set A is closed, convex, balanced, and contains a neighborhood of the origin. Such a set A is a unit ball of some vector norm $|\cdot|$ on matrices. ~~We~~ They then show that A is stable if and only if the above norm dominates the spectral radius ρ (rho). That is $\rho(A) \leq |A|$ (rho; TRUE VALUE OF A) for all matrices A. The necessity of the above condition is obvious, and is sometime referred to as the Neumann condition. To prove the sufficiency ~~we~~ the authors use the Kreiss matrix theorem and other results.

TRUE VALUE OF ·

ALL SPECTRAL DOMINANT NORMS ARE STABLE

S. Friedland* and C. Zenger**

1. Introduction

Let $\mathcal{A}$ be a set of $n \times n$ complex valued matrices - $M_n(C)$. $\mathcal{A}$ is called stable if

$$(1.1) \qquad |A^m| \le K, \quad m = 0,1,2,\ldots, \quad A \in \mathcal{A}.$$

Here $|\cdot|$ is a vector norm on $M_n(C)$.

In 1962 Kreiss [5] characterized stable sets. In particular he showed that (1.1) is equivalent to

$$(1.2) \qquad |(zI - A)^{-1}| \le C/(|z| - 1), \quad \text{for all } |z| > 1, \quad A \in \mathcal{A}.$$

While (1.1) easily implies (1.2) with $K = C$ it can be shown that (1.2) implies (1.1) with

$$(1.3) \qquad K = \alpha_n C, \ \alpha_n \le \frac{32en}{\pi}, \quad \lim_{n\to\infty} \alpha_n = \infty.$$

See for example [7] and [9].

The serious draw-back of (1.2) is that it is difficult to verify in general. Thus, a natural question is whether the condition (1.2) can be replaced by a simpler condition assuming the set $\mathcal{A}$ is of a certian type. In many instances $\mathcal{A}$ is of the following type:

(i) $\mathcal{A}$ - is closed,

(ii) $\mathcal{A}$ - is convex,

(iii) $\mathcal{A}$ - is circular, i.e. $e^{i\theta}\mathcal{A} = \mathcal{A}$ for all $\theta \in R$,

(iv) $\mathcal{A}$ - contains an open set.

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Clearly, these conditions are equivalent to the assumption that A is a unit ball of some vector norm $|\cdot|$ on $M_n(C)$,

$$A = \{A, |A| \le 1\} . \tag{1.4}$$

Thus, $|\cdot|$ is called stable if its unit ball is a stable set. For $A \in M_n$, let $\rho(A)$ denote the spectral radius of A. Since on finite dimensional vector space all the norms are equivalent we have the equality

$$\rho(A) = \lim_{m\to\infty} |A^m|^{1/m} . \tag{1.5}$$

So, if A is a stable set we get

$$\rho(A) \le 1, \quad A \in A . \tag{1.6}$$

Thus if $|\cdot|$ is a stable norm we have that $\rho(A) \le 1$ for $|A| = 1$. Using the homogeneity of $\rho(\cdot)$ and $|\cdot|$ we get

$$\rho(A) \le |A| . \tag{1.7}$$

Recall that $|\cdot|$ is called spectral dominant if (1.7) holds. Our main result is

Theorem 1. Let $|\cdot|$ be a vector norm on $M_n(C)$. Then $|\cdot|$ is stable if and only if it is spectrally dominant.

This result was conjectured by C. Johnson in [4]. The case of unitary invariant norms was proved in Friedland-Tadmor in [3].

2. Main Results

Following Zenger [10] we first consider special spectral dominant norms on $M_n(C)$. These norms are called the generalized numerical radius and are denoted by $r_g(\cdot)$. For reader's convenience we give short proofs of these known results. Let $\|\cdot\|_2$ be the standard Euclidean norm on C^n. As usual let x be a column vector in C^n, x^t and x^* its transpose and conjugate transpose. Denote by S_2 the unit sphere of this norm.

Assume that we have the following map

$$\phi : S_2 \to 2^{C^n} . \tag{2.1}$$

We suppose that

$$y^t x = 1, \quad \text{for all} \quad y \in \phi(x) . \tag{2.2}$$

We now assume that the map (2.1) is closed. That is, if $x_k \in S_2$, $y_k \in \phi(x_k)$, $x_k \to x$, the sequence $\{y_k\}$ is bounded. Moreover if $y_k \to y$ then $y \in \phi(x)$. This in particular implies that $\phi(x)$ is compact and $\bigcup_{x \in S_2} \phi(x)$ is bounded. We then define the generalized numerical radius as

$$r_g(A) = \max_{x \in S_2,\ y \in \phi(x)} |y^t A x| . \tag{2.3}$$

<u>Lemma 1</u>. <u>The generalized numerical range is a spectral dominant norm on</u> $M_n(C)$.

<u>Proof</u>: In view of (2.2) - (2.3) we have that

$$\rho(A) \le r_g(A) . \tag{2.4}$$

Also (2.3) yields that $r_g(A)$ is a seminorm. Assume that $r_g(A) = 0$ and $A \ne 0$. According to (2.4) A is nilpotent. Choose a basis in C^n such that A is of the form

$$A = \left(\begin{array}{cccc|c} 0 & 1 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & \ddots & 1 & \\ 0 & & & 0 & \\ \hline & & & & A_1 \end{array}\right) .$$

Let B be of the form

$$B = \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right) .$$

Then for $\epsilon > 0$

$$\rho(A + \epsilon B) = \sqrt{\epsilon} \le r_g(A + \epsilon B) = \epsilon r_g(B) .$$

Clearly this inequality can not hold for any $\epsilon > 0$. The above contradiction shows that $r_g(A)$ is a vector norm.

Let $\|\cdot\|$ be a norm on $X = \mathbb{C}^n$. Denote by $\|\cdot\|^*$ the corresponding norm of the dual space X^*,

$$\|y\|^* = \max_{\|x\| \le 1} |y^t x| .$$

We then let

$$\phi(x) = \{y,\ y^t x = 1\ \|y\|^* \|x\| = 1\},\quad (x \ne 0) .$$

It is easy to show that in this case the map (2.1) is closed. The corresponding generalized numerical radius is the Bauer numerical radius with respect to the norm $\|\cdot\|$. See [2] for details and references.

Denote by $\mathcal{U}$ the set of unitary matrices in $M_n(C)$. As usual by $\|\cdot\|_2$ we denote the induced operator norm on $M_n(C)$

$$\|A\|_2 = \max_{\|x\|_2 \le 1} \|Ax\|_2 . \tag{2.5}$$

<u>Theorem 2. Let $r_g(\cdot)$ be a generalized numerical radius. Put</u>

$$C = \max_{U \in \mathcal{U}} r_g(U) . \tag{2.6}$$

<u>Then</u>

$$\|(zI - A)^{-1}\|_2 \le C/(|z| - 1) \quad \text{for all} \quad |z| > 1,\ r_g(A) \le 1 . \tag{2.7}$$

<u>In particular a generalized numerical radius is a stable norm.</u>

<u>Proof</u>: We first note that

$$\|y\|_2 \le C \quad \text{for all} \quad y \in \phi(x) . \tag{2.8}$$

Indeed for $x \in B_2$ there exist $y \in$ [illegible] such that $y^t x = \|y\|_2$. Then (2.8) follows from (2.6). Assume next that $r_g(A) \le 1$. So $(zI - A)^{-1}$ is defined for $|z| > 1$. Let

$$v = (zI - A)^{-1}x, \; x \in B_2, \; v_1 = v/\|v\|_2 .$$

Then, for $y \in \phi(v_1)$ we have

$$\frac{|y^t x|}{\|v\|_2} = |y^t(zI - A)v_1| = |z - y^t A v_1| > |z| - 1 .$$

On the other hand

$$|y^t x| < \|y\|_2 \|x\|_2 < C .$$

Combine the above inequalities to get

$$\|(zI - A)^{-1}x\|_2 < C/(|z| - 1), \quad \|x\|_2 = 1 .$$

This proves (2.7). Now the stability of the generalized numerical radius follows from the Kreiss matrix theorem.

Finally Theorem 1 follows from Theorem 2 and Zenger's theorem [10] whose proof we bring for reader's convenience.

<u>Theorem 3</u>. (Zenger). <u>Let</u> $|\cdot|$ <u>be a spectral dominant norm on</u> $M_n(C)$. <u>Then there exist a generalized numerical radius</u> $r_g(\cdot)$ <u>which is subordinate to</u> $|\cdot|$. <u>That is</u>

(2.9) $$r_g(A) < |A| \quad \text{for all} \quad A \in M_n(C) .$$

<u>Proof</u>: Let $\mathcal{A}$ be the unit ball of $|\cdot|$. Consider the convex balanced set

$$\mathcal{A}_1 = \{B, B = (1 - \alpha)A + zI, \; z \in C, \; 0 < \alpha < 1, \; |z| < \alpha\} .$$

Clearly

$$\sigma(B) = (1 - \alpha)\sigma(A) + z$$

where $\sigma(B)$ is the spectrum of B. As $\rho(A) < 1$ for $A \in \mathcal{A}$ we have that

(2.10) $$\rho(B) < 1 \quad \text{for} \quad B \in \mathcal{A}_1 .$$

Also as $\mathcal{A} \subseteq \mathcal{A}_1$, $\mathcal{A}_1$ is the unit ball of a new norm $|\cdot|_1$ such that

(2.11) $$|A|_1 < |A| .$$

The inequality (2.10) implies that $|\cdot|_1$ is also spectral dominant. For $\|x\|_2 = 1$ let

(2.12) $$B(x) = \{u, u = Ax, \; |A|_1 < 1\} .$$

Clearly $B(x)$ is convex and $x \in \partial B(x)$. The separation theorem yields the existence of $y \in C^n$ such that

$$\mathrm{Re}(y^t x) \geq \mathrm{Re}(y^t Ax), \quad |A|_1 \leq 1 .$$

Since $|aI|_1 \leq 1$ for $|a| \leq 1$ we immediately deduce that $y^t x$ is real and positive. Thus we can normalize y such that $y^t x = 1$. Let $\phi(x)$ be the set of all w such that

$$w^t x = 1, \quad \max_{|A|_1 \leq 1} |w^t Ax| = 1 . \tag{2.13}$$

Clearly, $y \in \phi(x)$. As

$$w^t Ax = \mathrm{tr}(Axw^t) \tag{2.14}$$

we deduce that

$$|xw^t|_1^* = 1 \tag{2.15}$$

where $|\cdot|_1^*$ is the conjugate norm to $|\cdot|_1$ on $M_n(C)$

$$|B|_1^* = \max_{|A|_1 \leq 1} |\mathrm{tr}(AB)| . \tag{2.16}$$

This in particular implies that the map ϕ is closed. Since any finite dimensional vector space is reflexive we have that

$$|A|_1 = \max_{|B|_1^* \leq 1} |\mathrm{tr}(AB)| . \tag{2.17}$$

Compare (2.3), (2.14), (2.15) with (2.16) to deduce

$$r_g(A) \leq |A|_1 . \tag{2.18}$$

Then (2.11) implies (2.9).

3. Open Problems

Let $r(A)$ be the standard numerical radius of A

$$r(A) = \max_{\|x\|_2 = 1} |x^* A x| . \tag{3.1}$$

The Halmos inequality yields

$$r(A^m) \le r(A)^m . \tag{3.2}$$

See for example [8] for a short proof.

We call the unit ball $\mathcal{A}$ of a vector norm to be super stable, if

$$\mathcal{A}^m \subset \mathcal{A}, \qquad m = 1,2,\ldots \tag{3.3}$$

where

$$\mathcal{A}^m = \{B, B = A^m, A \in \mathcal{A}\} . \tag{3.4}$$

The inequality (3.2) implies that $r(\cdot)$ is a super stable norm. This in particular yields the Lax-Wendroff result [6] that the set $r(A) \le 1$ is stable. In fact, Theorem 1 is a natural extension of the Lax-Wendroff condition.

<u>Problem 1</u>. <u>Characterize all spectral dominant norms on</u> $M_n(C)$ <u>which are super stable</u>.

Clearly, the standard operator norm on $M_n(C)$ is super stable.

In many numerical schemes for solutions of partial equations one has to consider a stable set of matrices $\mathcal{A}$ whose order is not fixed, but in fact can be arbitrary large. In that case the Kreiss matrix theorem does not apply. See for example [7]. Therefore one needs to study stable sets in the infinite dimensional case. Let B be a Banach space with a norm $\|\cdot\|$, $L(B)$ the space of all linear bounded operators $T : B \to B$ with the induced operator norm $\|\cdot\|$. Assume that $|\cdot|$ is a norm on $L(B)$ which is equivalent to the operator norm

$$\alpha|T| \le \|T\| \le \beta|T|, \qquad 0 < \alpha < \beta . \tag{3.5}$$

As before, we call $|\cdot|$ stable if the unit ball of this norm is stable, i.e., (1.1) holds.

and $\rho(B)$ is the spectral radius of B is

$$\rho(B) = \lim_{k\to\infty} \|B^k\|^{1/k} \tag{3.6}$$

the inequality (3.4) yields that a stable norm is spectral dominant.

<u>Problem 2</u>. <u>Characterize all norms on</u> $L(B)$ <u>which are stable and equivalent to the operator norm</u>.

We note that there are spectral dominant norms on $L(B)$ which are not stable and are equivalent to the operator norm. Indeed, if we choose $|B|$ to be the numerical radius of B with respect to the given norm $\|\cdot\|$ we then have the inequalities

$$|B| \le \|B\| \le e|B| .$$

See for example [2]. Furthermore, according to Bollobás [1], Theorem 2, there exists an operator B such that $|B| = 1$, $\|B\| = e$, $\|B^k\| > \sqrt{k}$, $k = 2,3,\ldots$.

Finally we close our paper with a very specific problem. For $x = (x_1,\ldots,x_n)^t \in C^n$ let

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \qquad 1 < p < \infty$$

(3.7)

$$y(x) = (\bar{x}_1|x_1|^{p-2},\ldots,\bar{x}_n|x_n|^{p-2})^t, \qquad x \neq 0 .$$

Then, for $A \in M_n(C)$ we define $r_p(A)$ - the p-th numerical radius

$$r_p(A) = \max_{\|x\|_p=1} |y^t(x)Ax| . \tag{3.8}$$

Theorem 1, in this case is equivalent to the inequality

$$r_p(A^m) \le K_{p,n} r(A)^m, \qquad m = 0,1,2,\ldots , \tag{3.9}$$

for all $A \in M_n(C)$. The inequality (3.2) yields that

$$K_{2,n} = 1 . \tag{3.10}$$

We may assume in (3.9) that $K_{p,n}$ is best possible. In that case clearly

$$K_{p,n} \le K_{p,n+1} . \tag{3.11}$$

Let q be conjugate to p

$$p^{-1} + q^{-1} = 1 . \tag{3.12}$$

[illegible]

[illegible]

[illegible]

[illegible]

Problem 1. [illegible] p, $1 < p < \infty$, [illegible] is bounded?

We considered the extreme cases $p = 1, \infty$. [illegible] that one can define $r_p(A)$ in a similar way for $p = 1, \infty$. Also we can let

(3.15) $$r_1(A) = \lim_{p \to 1^+} r_p(A), \quad r_\infty(A) = \lim_{p \to \infty} r_p(A) .$$

Let $\|A\|_p$ be the induced operator norm on $M_n(C)$. Then it is easy to show that

(3.16) $$r_1(A) = \|A\|_1, \quad r_\infty(A) = \|A\|_\infty .$$

So

(3.17) $$K_{1,n} = K_{\infty,n} = 1 .$$

The equalities (3.10) and (3.17) suggest that $(K_{p,n})_{n=1}^{\infty}$ is always bounded.

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